ALGEBRAIC CURVES
EXERCISE SHEET 13

Unless otherwise specified, k is an algebraically closed field of characteristic 0.

Exercise 1. Let $a \in k^*$ and consider the elliptic curve E with equation

$$X^3 + Y^3 = aZ^3,$$

and base point $O = [1, -1, 0]$.

(1) Prove that three points on E add to O if and only if they are collinear.
(2) Let $P = [x : y : z] \in E$. Prove $-P = [y : x : z]$.
(3) Prove that E has j-invariant 0.

Exercise 2. Let $O = [0 : 1 : 0]$ be a flex on an irreducible cubic F and $Z = 0$ the tangent line to F at O.

(1) Show that $F = ZY^2 + bYZ^2 + cYXZ +$ terms in X, Z.
(2) Find a projective change of coordinates (using $Y \mapsto Y - \frac{b}{2}Z - \frac{c}{2}X$) to get F to the form

$$ZY^2 = \text{ cubic in } X, Z.$$

(3) Show that any non-singular cubic is projectively equivalent to

$$Y^2Z = X(X - Z)(X - \lambda Z),$$

for a $\lambda \in k$, $\lambda \neq 0, 1$. This is called the Legendre form of an elliptic curve.

Exercise 3.

(1) Use Ex. 5.4 to show that given two triples (p_1, p_2, p_3) and (q_1, q_2, q_3) each of distinct points in $\mathbb{P}^1$ there exists a unique projective change of coordinates sending p_i to q_i for $i = 1, 2, 3$.
(2) The *cross-ratio* of four distinct ordered points (p_1, p_2, p_3, p_4) in $\mathbb{P}^1$ is defined as $\lambda \in k \setminus \{0, 1\}$, where λ is the image of p_4 under the unique projective change of coordinates sending (p_1, p_2, p_3) to $(\infty, 0, 1)$.

(3) Show that this defines an action of S_3 on $k \setminus \{0,1\}$ and the orbit $\mathcal{O}_\lambda$ of $\lambda \in k \setminus \{0,1\}$ is

$$\mathcal{O}_\lambda = \{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), (\lambda-1)/\lambda, \lambda/(\lambda-1)\}.$$

Exercise 4. Let E_λ be an elliptic curve given in its Legendre form

$$Y^2 = X(X-1)(X-\lambda),$$

with $\lambda \neq 0, 1$.

(1) Show that the j-invariant is given by

$$j(E_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}.$$

(2) Show that $E_\lambda \cong E_\mu$ if and only if $\mu \in \mathcal{O}_\lambda$.

In fact one can use (2) to find the formula for the j-function, as it is a generator of the fixed field $k(\lambda)^{S_3}$.